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Research Article

On the Numerical Solution for Two Dimensional Laplace Equation with Initial Boundary Conditions by using Finite Difference Methods

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Abstract

In this study, Laplace partial differential equations with initial boundary conditions has been studied. A numerical method has been proposed for the solution of the IBVP Laplace equation. The technique based on finite difference methods. The stability of the difference schemes are guaranteed. Approximation solution of the problem was achieved. For testing the accuracy of the proposed method, two different initial boundary value problems are provided. Moreover, a comparison between the numerical solution and analytical solution has been done. MATLAB software implemented for calculation of absolute errors. Illustration graphs presented. It has been demonstrated that the results of the comparison guarantee the accuracy and reliability of the provided method.

1. Introduction

Recently, Differential Equations (DE) have been the focus of a lot of studies [1, 2]. DE is important for modeling various problems such as electrodynamics, elasticity, wave propagation, signal analysis, and thermodynamics [3-5]. Differential equations have been applied for many problems in science, engineering, finance, physics and seismology[1, 6-9]. Many methods for the exact solution have been presented for solving Differential equations[10-14]. Although the exact solution of some types of DE can be found [1-3, 5]., many numerical methods have been applied for solving linear and non-linear differential equations [15, 16]. In the case of Partial differential equations, much attention has been given in the works to test the reliability and accuracy of the approximation and numerical techniques [17]. Many researchers applied finite difference approximations for numerical solution of different kinds of PDE's in works [7, 18-25].

Qian and Weiss are studies Wavelets and the numerical solution of PDE's [22]. Ashyralev and Modanli presented finite difference method for approaching numerical solution for telegraph PDE [26]. Soleymani and Akgül worked on numerical solution of multi-asset option pricing problem [19]. Su et al., presented a numerical solution for fractional advection–diffusion equation, they used Finite difference approximations in their works [23]. Akgul and Faraj have applied Finitie difference methods for Wave equations under IBV conditions [18]. Moreover, various studies on PDE stability estimates have been done, scholars presented several methods for proving the stability of the proposed methods [7, 19-33]. In the other hand, various numerical methods presented for solving IBVP elliptic partial differential equations [26, 28-30].

2. The Laplace Equation

Numerous physical and engineering phenomena are modeled using the Laplace equation with boundary condition. Consequently, a great deal of focus has been given to examining this problem. Some theoretical and numerical treatments of this problem are found in [20, 34]. Finite difference methods are the oldest numerical methods for solving this equation [21]. In the other hand, Al-Haq and Mohammad in [28] investigated the method of fundamental solutions for solving the Laplace equation with the Dirichlet boundary condition in a disk[35]. Moreover, several different numerical techniques were presented for solving Laplace equations [17, 27, 34, 35]

It is possible to approach the derivation of Laplace's equation by making use of a number of physical processes associated to science and engineering. However, in the following, we'll concentrate on the equation's derivation by taking applications in the general field of fluid mechanics into account [36].

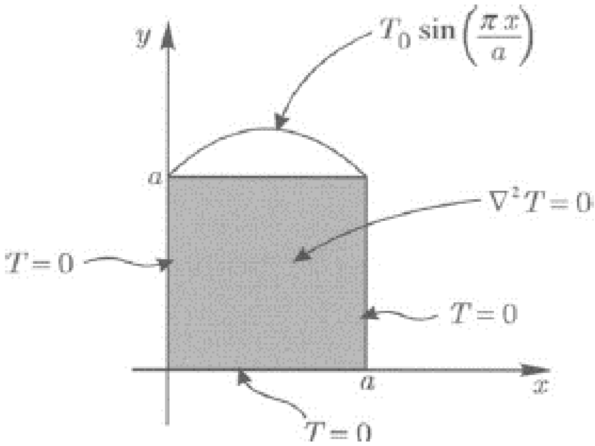

Figure 1. Plate subjected to boundary temperature

Solutions to Laplace's equation in two and three dimensions are known as harmonic functions. The inhomogeneous version of Laplace's equation

$$\nabla^2 u = f,$$

is called Poisson's equation. This equation has been extensively studied by mathematicians and physicists [5, 16, 37]. In the present study, we consider Laplace equation under initial boundary conditions, as follow:

$$\begin{cases} u_{tt}(t,x) + u_{xx}(t,x) = 0 & 0 \le x \le L, \\ u(t,0) = u(t,\pi) = 0, & 0 \le t \le T, \\ u(0,x) = \psi(\mathrm{x}), \mathrm{u_t}(0,x) = \varphi(x). \end{cases} \tag{1}$$

Where $\varphi(x), \psi(x)(x \in [0,\pi])$ and $f\ (t,x) = 0\ ,((t,x)\ \in\ [0,1]\ \times\ [0,\pi])$ are smooth functions. Problem (1) presents a IBVP Laplace equation [7].

3. Proposed Method

In this section we introduce the explicit and implicit methods for approximate solution of IBVP Laplace equation in (1). By using central differences for both time and space derivatives, we obtain

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = 0 \tag{2}$$

The difference equation in (2) called first order difference scheme. By rewritten (2) in terms of u_n, we get

$$\left(\frac{1}{h^2}\right) u_{n+1}^k + \left(\frac{1}{\tau^2}\right) u_n^{k+1} - 2\left(\frac{1}{\tau^2} + \frac{1}{h^2}\right) u_n^k + \left(\frac{1}{\tau^2}\right) u_n^{k-1} + \left(\frac{1}{h^2}\right) u_{n-1}^k = 0 \tag{3}$$

By supposing that $a = \frac{1}{h^2}$, $b = \frac{1}{\tau^2}$ and putting in (3), we obtain

$$a u_{n+1}^k + b u_n^{k+1} - 2(a+b) u_n^k + b u_n^{k-1} + a\, u_{n-1}^k = 0 \tag{4}$$

Equation (4) can be transformed to matrices [21].

For the problem (1), we shall present introduce an implicit scheme from (2). By replacing all terms on $\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2}$ of (2) by an average from the values to the time steps $k-1$ and $k+1$, we obtain

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \frac{1}{2h^2}\left(u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1} + u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}\right) = 0 \tag{5}$$

The difference equation in (5) is second order difference scheme for the problem (1), we used implicit method. The stability of the difference scheme's in (2) and (5) is guaranteed in [38-40]. By rewritten (5) in terms of u_n, we obtain:

$$\left(\frac{1}{2h^2}\right) u_{n+1}^{k+1} + \left(\frac{1}{2h^2}\right) u_{n+1}^{k-1} + \left(\frac{1}{\tau^2} - \frac{1}{h^2}\right) u_n^{k+1} + \left(\frac{-2}{\tau^2}\right) u_n^k + \left(\frac{1}{\tau^2} - \frac{1}{h^2}\right) u_n^{k-1} + \left(\frac{1}{2h^2}\right) u_{n-1}^{k+1} + \left(\frac{1}{2h^2}\right) u_{n-1}^{k-1} = 0$$

By putting $a = \frac{1}{2h^2}$, $b = \frac{1}{\tau^2} - \frac{1}{h^2}$, $c = \frac{-2}{\tau^2}$ and in above equation, we obtain

$$au_{n+1}^{k+1} + au_{n+1}^{k-1} + bu_n^{k+1} + c\,u_n^k + bu_n^{k-1} + au_{n-1}^{k+1} + au_{n-1}^{k-1} = 0 \qquad (6)$$

Both difference equations in (3) and (6) can be rewritten to the following matrices' formulas [21] are obtained as

$$AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, u_0 = u_M = 0. \qquad (7)$$

where, A, B and C are $(N+1) \times (N+1)$ matrix, U_{n+1}, U_n, U_{n-1} and φ_n is $(N+1) \times 1$ vectors, in the present work $\varphi_n = 0$. [7, 21] we applied an operation of modified Gauss elimination method for solving the difference equation in (7). Then, we are looking for a matrix equation solutions, which was presented as follows [26]:

$$u_j = \alpha_{j+1} u_{j+1} + \beta_{j+1};\ u_M = 0;\ j = M-1, \dots, 2, 1.$$

Where β_j are $(N+1) \times 1$ column vectors, and α_j are $(N+1) \times (N+1)$ square matrices, defined by

$$\alpha_{j+1} = -(B + C\alpha_j)^{-1} A,$$

and

$$\beta_{j+1} = (B + C\alpha_j)^{-1}(D\varphi - C\beta_j), \quad j = 1, 2, \dots, M-1,$$

where $j = 1, 2, \dots, M-1$, β_1 is the $(N+1) \times 1$ zero column vector, and α_1 is the $(N+1) \times (N+1)$ zero matrix. The results are computed by Matlab program. For three distinct values of M and N, the comparison was made between numerical and exact solution.The maximum error indicated where $h = \pi/M$, and $\tau = 1/N$. Maximum absolute error calculated by

$$E_M^N = \max_{\substack{1 \le k \le N-1 \\ 1 \le n \le M-1}} |u(t_k, x_n) - u_n^k|,$$

where exact solution denoted by $u(t_k, x_n)$ and approximation solution denoted by u_n^k at points (t_k, x_n).

Remark 1. The stability of the presented method have been instigated in [7, 18, 21, 26]

4. Numerical Experiments

In this section, we will apply first order difference scheme in (2) and second order difference scheme in (5) for two different numerical examples for the Laplace PDE with initial-boundary conditions. The ability of the procedure will be evaluated by comparing the exact solutions for both examples to the numerical result. In this paper we consider the following Laplace initial boundary value problem:

$$\begin{cases} u_{tt}(t,x) + u_{xx}(t,x) = 0 & 0 \le x \le L, \\ u(t,0) = u(t,\pi) = 0, & 0 \le t \le T \\ u(0,x) = \psi(x), u_t(0,x) = \varphi(x) \end{cases}$$

Example 1. Consider the following Laplace equation with initial boundary value condition

$$\begin{cases} u_{tt} + u_{xx} = 0 & 0 \le x \le \pi, 0 \le t \\ u(t,0) = u(t,\pi) = 0 \\ u(0,x) = \sin(x), u_t(0,x) = -\sin(x) \end{cases} \tag{8}$$

Analytical solution of the problem (8) can be found by using one of the exact methods, which is $u(t,x) = \frac{\sin(x)}{e^t}$. Then the first order difference scheme for (8) is presented as follow:

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = 0 \\ u_0^k = u_\pi^k = 0 \quad 0 \le n \le N, 0 \le m \le M \\ u_n^0 = \sin(x_n)\ , \dfrac{u_n^1 - u_n^0}{\tau} = -\sin(x_n) \end{cases} \tag{9}$$

For second order difference scheme, we obtain

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \dfrac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1} + u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{2h^2} = 0 \\ u_0^k = u_\pi^k = 0 \quad 0 \le n \le N, 0 \le m \le M \\ u_n^0 = \sin(x_n)\ , \dfrac{u_n^1 - u_n^0}{\tau} = \dfrac{\tau}{2}\dfrac{u_n^2 - 2u_n^1 + u_n^0}{\tau^2} - \sin(x_n) \end{cases} \tag{10}$$

The error analyze of problem (8) shown in table 1, however the figures of exact and approximation solution has been shown in figure 2.

Example 2. Investigate the following IBVP Laplace PDE

$$\begin{cases} \nabla^2 u = 0 & 0 \le x \le \pi, 0 \le t \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin(x), u_t(x,0) = \sin(x) \end{cases} \tag{11}$$

Exact solution of the problem is $u(x,t) = \sin(x)\exp(t)$. First order difference scheme for problem (11) is

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = 0 \\ u_0^k = u_\pi^k = 0 \quad 0 \le n \le N, 0 \le m \le M \\ u_n^0 = \sin(x_n), \dfrac{u_n^1 - u_n^0}{\tau} = \sin(x_n) \end{cases} \tag{12}$$

Second order difference scheme (Implicit method) can be obtained as follow:

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}+u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2}=0 \\ u_0^k=u_\pi^k=0, \qquad 0\le n\le N, 0\le m\le M \\ u_n^0=\sin(\mathrm{x_n}), \qquad \dfrac{u_n^1-u_n^0}{\tau}=\dfrac{\tau}{2}\dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2}+\sin(x_n) \end{cases} \tag{13}$$

The error analyzes of the problem (11) is shown in table 2, however, the figures of exact and approximation solution have been shown in figure 3.

Table 1. Absolute Error for the Difference schemes in (9) and (10) in different values of N and M

$h=\pi/M,\tau=1/N$	N=M=15	N=M=30	N=M=45
First Order Difference in (9)	3.9515×10^{-2}	1.9751×10^{-2}	1.4115×10^{-2}
Second Order Difference in (10)	2.3725×10^{-3}	6.0666×10^{-4}	2.1278×10^{-2}

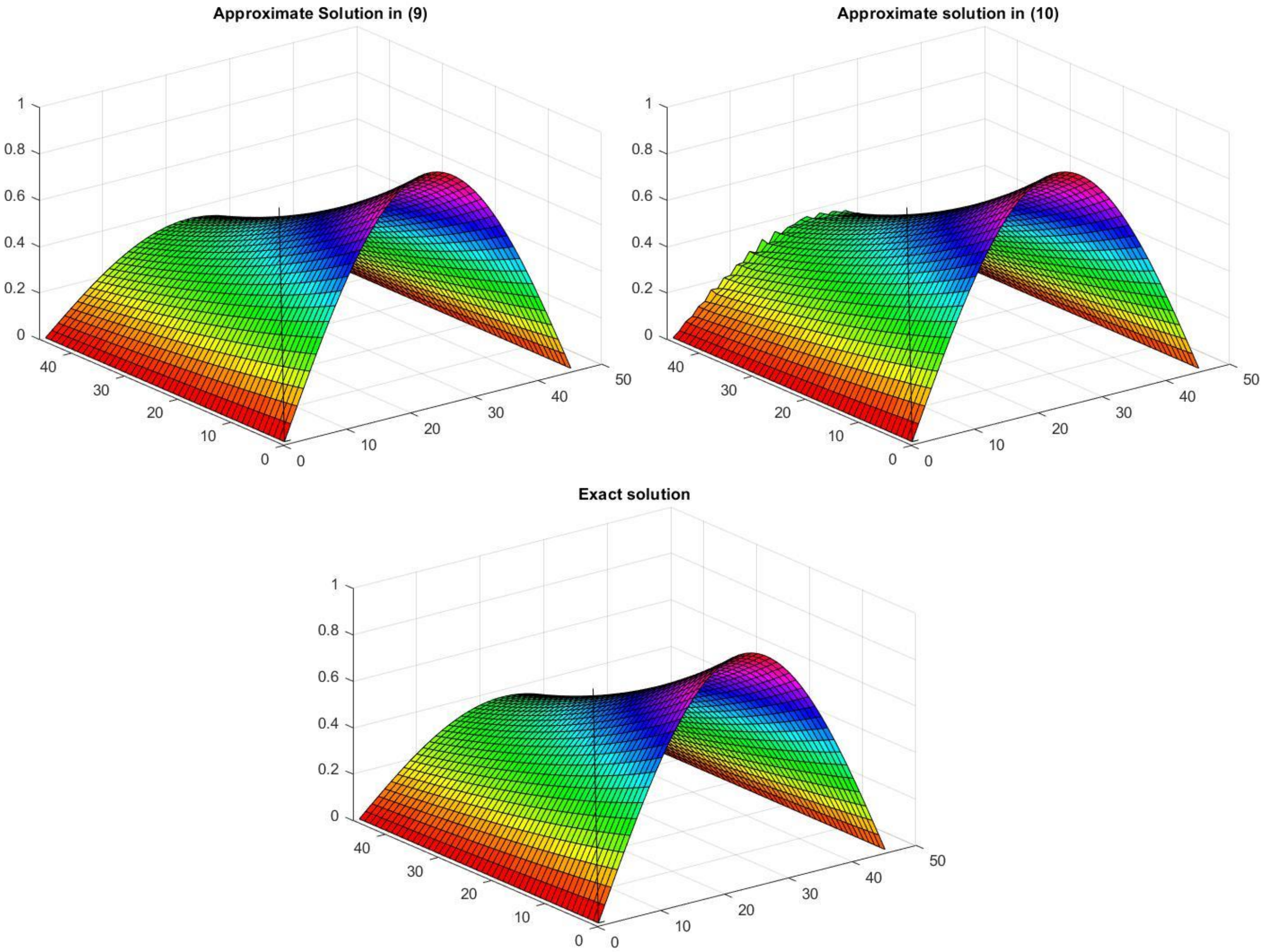

Figure 2. Exact solution and numerical solution for the problem (8) where N=M=45.

Table 2. Absolute Error for the Difference schemes in (12) and (13) in different values of N and M

$h = \pi/M, \tau = 1/N$	N=M=15	N=M=30	N=M=45
First Order Difference in (12)	4.2707×10^{-2}	2.0555×10^{-2}	1.4857×10^{-2}
Second Order Difference in (13)	5.8787×10^{-4}	1.4119×10^{-4}	4.4080×10^{-2}

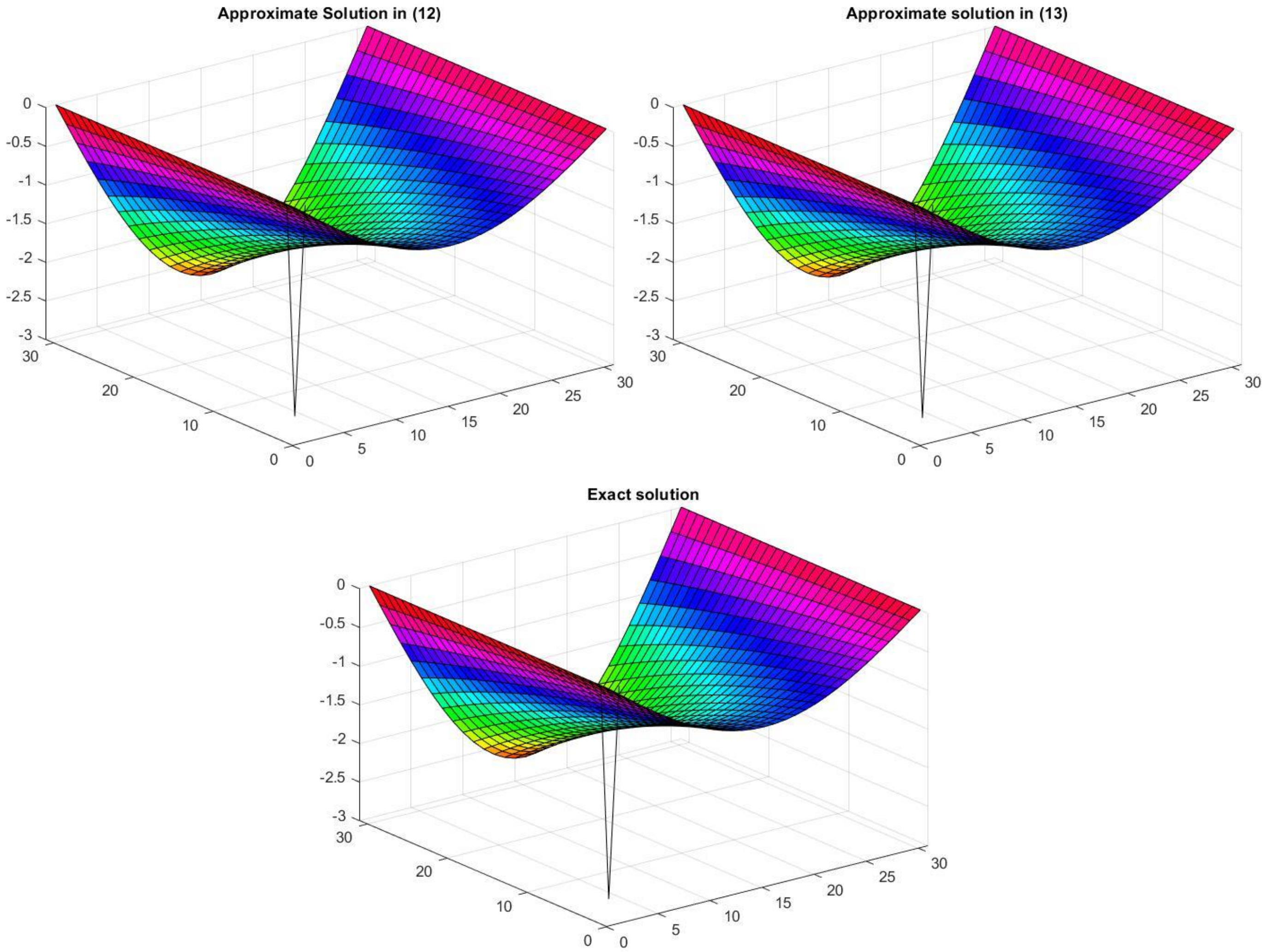

Figure 3. Exact solution and numerical solution for the problem (11) where N=M=30.

From Table 1 and Table2 we conclude that the method has accuracy and applicable for Laplace equations. We note that the second order difference scheme has more accuracy than the first order.For the first order difference scheme, the results are accurate when N=M=45. In the other hand, in Second order difference scheme, Results will be accurate in case N=M=30. By comparing both tables, we achieve that the method has accurate results when applied for Laplace equation.

Remark 2. The present method is applicable for Laplace equation under IBV conditions, in small values of N and M. When the values of N and M exceed 50, the singular matrix will appear in calculations because $(f(t,x)=0)$; hence results will be less accurate. The proposed method can be applied for the Laplace equation in values of N,M less than 50.

5.Conclusion

In this study, Laplace equation under Initial boundary conditions have been investigated. First order and second order difference schemes for the Laplace partial differential equations are showed. Stability for the problem and difference scheme has been guaranteed. Modified Gauss elimination method used for calculation numerical solution MATLAB software implemented for numerical calculation. Two Laplace IBVP with exact solution presented. Proposed method applied for both problems. The numerical and analytical solution of the problem were compared to obtain the error analysis in the maximum norm. Results shows that the proposed method is suitable for the Laplace equation under IBV conditions. Second order difference scheme is more accurate than first order difference scheme. Moreover, the proposed method is suitable for Laplace equation in step size less than 50.

Declaration of Competing Interest The authors declare that they have no known competing of interest.

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